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INVERSE PROBLEMS IN PRICING FINANCIAL DERIVATIVES AND THEIR REGULARIZATION FOR THE AMERICAN OPTION CASE

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INTRODUCTION

The volatility identification problem for generalized Black-Scholes models is an ill-posed problem [4]. From a financial application point of view, the problem can be viewed as market price calibration.

During recent years, several approaches have been proposed to this problem solution (for example, [3]), mainly in the context of European Vanilla options. In this contribution, we are interested in the solution of the volatility identification problem for American options. Since typical procedures for reconstructing local volatilities are unstable with respect to small perturbations of the data, we apply the Tikhonov regularization technique to solve the problem computationally. In contrast to European options, in the case of American options we have an additional freedom of being able to exercise the option at any time during the contract timeframe. From a mathematical point of view this brings additional difficulties in the development of efficient computational procedures. However, from a financial point of view, this brings a possibility of early-exercise premiums, and we discuss the difference between the two cases with numerical examples.

MATHEMATICAL MODEL

The partial differential equations (PDE) framework for the option pricing problems considered in this contribution is based on the Black-Scholes model. Recall that in the European option case, the situation can be cast in the Black-Scholes framework as follows. If U is the price of a European option at time t, the spot price is S, and the strike price is K, then we can determine $U(t,S)$ by solving the following PDE (in the no-divident case):

$$\frac{\partial U}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 U}{\partial S^2}+rS\frac{\partial U}{\partial S}-rU=0, \tag{1}$$

where r is the interest rate and the terminal (final) condition is given by the payoff function, i.e.,

$$U(T,S)=\max(K-S,0) \tag{2}$$

for a put option. At the boundaries we have:

$$U(0,t)=K \tag{3}$$

$$U\to 0 \quad \text{as} \quad S\to\infty. \tag{4}$$

Unlike European options (e.g., [2]), American options yield more complicated situations.

American Options

First, note that American options grant the holder the additional freedom to exercise the option at any time prior to the expiry of the contract. With the introduction of the possibility of early exercise, the direct problem for the price of an American option (the problem of pricing the American put option) becomes a free boundary problem. It can be formalized as:

$$\frac{\partial U}{\partial t}-\frac{\sigma^2 S^2}{2}\frac{\partial^2 U}{\partial S^2}-rS\frac{\partial U}{\partial S}+rU\geq 0 \tag{5}$$

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$$U(t,S) \geq U(0,S) \quad (6)$$

$$\left(\frac{\partial U}{\partial t} - \frac{\sigma^2 S^2}{2}\frac{\partial^2 U}{\partial S^2} - rS\frac{\partial U}{\partial S} + rU\right)(U(t,s) - U(0,S)) = 0 \quad (7)$$

$$U(0,S) = max(K-S,0), \quad (8)$$

where σ is the volatility function and $U(t,S)$, as before, is the option value.

The standard approach for the European option case would be to use the Dupire equation that would allow us to determine a unique local volatility function that made the underlying process fit the smile. In the case of American options, however, we cannot follow the same path in determining the local volatility because of the nonlinearity of the above problem.

INVERSE PROBLEM

The inverse problem of option pricing seeks to determine the volatility of the option price given the value of the option for known strike prices and expiry times. From a financial point of view this may be viewed as a model calibration problem, where the volatility parameter of the model must be fitted to market data. Within the framework of the Black-Scholes model, it is typically assumed that the local volatility takes the form of either a deterministic function of time and stock price, or is itself a stochastic process (e.g., [3,5]). It is known that the problem of reconstructing such a local volatility function is an ill-posed problem (e.g., [4]). Several different approaches have been proposed to the solution of this problem (see, for example, [3] and the references contained therein). These include solving a linearized version of the problem; approaching the problem as an optimal control problem and solving it via dynamic programming; or, solving the problem by applying a regularization technique. It is the latter approach which we explore further in this contribution. In [1], the authors apply the Tikhonov regularization technique for solving the inverse problem of American option pricing. They demonstrated that the algorithm is effective. In our contribution we apply the Tikhonov regularization approach and solve the problem numerically.

METHODOLOGY OF THE SOLUTION

The approach developed in this contribution is applied to reconstructing the local volatility function and is based on the solution of the following problem. Starting from an *a priori* guess η^*, we seek to find a positive and bounded function η such that

$$J(\eta) \to \min, \quad (9)$$

where

$$J(\eta) = ||u_i(\eta) - U_i||^2_{L^2} + \beta||\eta - \eta^*||^2_{H^1}. \quad (10)$$

Here $\{U_i\}$ is a given set of prices of American put options with different maturities and strikes (T_i, K_i), $u_i(\eta)$ solves the forward problem (5)–(8), and β is the regularization parameter. We develop a procedure to solve this problem numerically and analyze the computational efficiency of the developed procedure for different choices of the regularization parameter and the optimization algorithm used to perform the minimization step. The results are demonstrated with several numerical examples.

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